

**On spacetimes with 3-parameter isometry group in string-inspired  
theory of gravity**

P. Klepáč

*Institute of Theoretical Physics and Astrophysics, Faculty of Science,  
Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic*

E-mail: klepac@physics.muni.cz

Cylindrically symmetric stationary spacetimes are examined in the framework of string-inspired generalized theory of gravity. In four dimensions this theory contains a dilatonic scalar field in addition to gravity. A charged perfect fluid representing fermionic matter is also considered. Explicit solution is given and a discussion of the geometrical properties of the solutions found is carried out.

Key words: cylindricall symmetry, exact solutions, first-order correction

MS 2000 classification: 83C22, 83C15, 83E30

# 1 Introduction

In this work some results on the stationary cylindrically symmetric spacetimes in string-inspired generalized theory of gravity are presented. The reason for studying this class of the spacetimes is twofold. First, searching for stringy solutions is important in itself [1, 7]. Second, in classical relativity theory the cylindrically symmetric spacetimes are known to violate some of the chronology conditions [2]. Therefore it is natural to address the question of chronology violation in the stringy spacetimes. Barrow and Dąbrowski [1] had given string Gödel-type solutions and Kanti and Vayonakis [3] found the Gödel-type homogeneous metrics in the string-inspired charged gravity.

The paper is organized as follows. In Section 2 we consider an exact solution of a general relativistic system of a perfect fluid and a scalar field coupled to gravity. In Section 3 the results of the previous Section generalized on the  $\alpha'$ -order corrections in the framework of the string-inspired theory [7, 3]. Finally in Section 4 we treat some of the geometrical properties of the found solution.

# 2 Classical solution

Let spacetime  $(\mathcal{M}, \mathbf{g})$  be connected 4-dimensional smooth orientable manifold endowed with lorentzian metric  $\mathbf{g}$ . We search for cylindrically symmetric stationary spacetimes. Then there exist local coordinate systems  $(x^0, x^1, x^2, x^3) = (t, \varphi, z, r)$  adapted to Killing fields  $\partial_t, \partial_\varphi, \partial_z$ , where the hypersurfaces  $\varphi = 0$  and  $\varphi = 2\pi$  are to be identified and  $\partial_t$  is an everywhere nonvanishing timelike field.

Metric tensor field is expressed in the following way

$$\mathbf{g} = \eta_{\mu\nu} \Theta^\mu \otimes \Theta^\nu ,$$

where  $(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$  is Minkowski matrix and  $\Theta^\mu$  are local coframe fields (greek indices run from 0 to 3) that form a (pseudo-)orthonormal basis in each cotangent space and are defined by

$$\begin{aligned} \Theta^0 &= dt + f d\varphi , & \Theta^1 &= l d\varphi , \\ \Theta^2 &= e^\gamma dz , & \Theta^3 &= e^\delta dr , \end{aligned} \tag{2.1}$$

with  $f, l, \gamma, \delta$  being functions of  $r$  only. Through the text the Einstein summation rule is used.

Our starting action is

$$S[\mathbf{g}, \phi, \mu] = \int_{\mathcal{M}} *R - d\phi \wedge *d\phi + 16\pi * \mu , \tag{2.2}$$

where  $R$  is the scalar curvature of the metric tensor  $\mathbf{g}$ ,  $\phi(r, z)$  and  $\mu(r)$  are scalar fields on  $\mathcal{M}$ . From the physical point of view a coupled system of massless scalar

field  $\phi(z, r)$  and perfect fluid is considered. The fluid moves in its comoving system with velocity vector field  $\mathbf{u} = \Theta^0$  and it is characterised by the pressure  $p(r)$  and the energy density  $\mu(r)$ . (We use the same symbol for vector fields and their naturally corresponding 1-forms.)

The equations of motion for the metric field  $\mathbf{g}$  are the Einstein field equations in basis (2.1) written as [10]

$$-\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi * i_{\alpha} \mathbf{T} , \quad (2.3)$$

where

$$\eta^{\alpha\beta\gamma} = *(\Theta^{\alpha} \wedge \Theta^{\beta} \wedge \Theta^{\gamma}) ,$$

$\Omega$  is curvature 2-form on  $T\mathcal{M}$ ,

$$\Omega_{\beta}^{\alpha} = \frac{1}{2} R^{\alpha}_{\beta\gamma\delta} \Theta^{\gamma} \wedge \Theta^{\delta} ,$$

and  $\mathbf{T}$  is the stress-energy tensor

$$8\pi \mathbf{T} = 8\pi [(\mu + p)\mathbf{u} \otimes \mathbf{u} - p \mathbf{g}] + \frac{1}{2} \left[ d\phi \otimes d\phi - \frac{1}{2} \mathbf{g}(d\phi, d\phi) \mathbf{g} \right] . \quad (2.4)$$

The appropriate explicit form of the Einstein equations for the basis (2.1) is similar to the one in [4] so we simply refer the reader to the paper.

Bianchi identity  $D * i_{\alpha} \mathbf{T} = 0$  in our case implies  $p = \text{const}$  because of the fluid particles geodesic motion (see below).

Altogether we obtain five independent equations for six unknown functions:  $f$ ,  $l$ ,  $\gamma$ ,  $\delta$ ,  $\mu$  and  $\phi$ . Therefore the mathematical problem admits one degree of freedom corresponding to the  $r$ -coordinate rescaling possibility as can be seen directly from (2.1).

The scalar field equation of motion is the massless Klein-Gordon equation

$$* d * d \phi = 0 . \quad (2.5)$$

Because of the independence of  $r$  and  $z$  coordinates one can carry out the separation of variables in (2.5) to obtain

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int l e^{\delta - \gamma} dr , \quad (2.6)$$

where  $\phi_0$ ,  $\phi_1$  and  $\phi_2$  are constants such that  $\phi_1 \cdot \phi_2 = 0$ .

For the function  $l^2$  one gets a second-order non-linear equation that can be transformed into the following form

$$\frac{d}{dx} \left[ \frac{y' + x}{y} \right] = -\frac{k\phi_2^2}{y^2} , \quad k = \text{const} . \quad (2.7)$$

We proceed further by dividing solutions of (2.7) into two groups according as  $\phi_2$  is zero (*Case (a)*) or is not (*Case (b)*).

*Case (a):  $\phi_2 = 0$*

From (2.6) we can see that  $\phi$  depends linearly on  $z$  alone. The solution of the Einstein equations can be settled in the form

$$\mathbf{g} = dt \otimes dt + \frac{\Omega}{C} \gamma (dt \otimes d\varphi + d\varphi \otimes dt) - l^2 d\varphi \otimes d\varphi - e^{2\gamma} dz \otimes dz - C^{-2} l^{-2} de^\gamma \otimes de^\gamma . \quad (2.8)$$

The metric function  $l^2$  is given by

$$l^2 = \frac{8\pi p}{C^2} e^{2\gamma} - \frac{4\Omega^2 + \phi_1^2}{2C^2} \gamma + \nu , \quad (2.9)$$

with  $\Omega$ ,  $\nu$ ,  $C$  as integration constants and  $\gamma$  an arbitrary non-constant  $C^2$  function.

*Case (b):  $\phi_2 \neq 0$*

Now we have dust distribution ( $p = 0$ ) that generalizes the van Stockum solution and the explicit form of the metric is omitted here for the reason given below. The scalar field is integrated to

$$e^\phi = (a\tilde{\gamma} + b)^{\frac{\phi_2}{a}} ,$$

with  $\tilde{\gamma}$  being an arbitrary function and  $a$  and  $b$  constants.

In this work we focus on the spacetimes with 3-parameter isometry group, especially on the stationary cylindrically symmetric ones. If one requires the found solutions to be cylindrically symmetric regular at the origin the axial symmetry condition and the elementary flatness condition have to be imposed [5]. For *Case (a)* these conditions yield

$$8\pi p - \Omega^2 - \frac{1}{4}\phi_1^2 = C, \quad 8\pi p + C^2\nu = 0 .$$

On the hand it turns out that *Case (b)* cannot be cylindrically symmetric.

For further investigation of the charged scalar field and allowed frequencies in Gödel-type background we refer the reader to [6], where is also considered the problem of field quantization using the Euclidean approach to quantum field theory.

### 3 String-inspired theory

We consider a generalized theory of gravity which describes the coupling of a dilaton scalar field to an electromagnetic field and gravity with the following  $\alpha'$ -order

corrected effective action in the Einstein frame [3]

$$S_{\text{eff}}[\mathbf{g}, \mathbf{A}, \phi, \mu] = \int_{\mathcal{M}} \left[ *R + 16\pi * \mu - d\phi \wedge *d\phi + 2\lambda e^{\phi} \mathbf{F} \wedge *\mathbf{F} \right] . \quad (3.1)$$

Here  $\mathbf{F}$  is the electromagnetic field 2-form. Coupling constant  $\lambda$  is expressed like  $\lambda = \frac{\alpha'}{4g^2}$ , where  $\alpha'$  is the inverse string tension (Regge slope) and  $g$  is essentially the string coupling constant. Physically, if the fluid particles are charge carriers, the charged perfect fluid is rough approximation of a fermionic matter in the theory.

We are interested in  $\alpha'$ -order corrections to the classical solution (2.8). It means that in (3.1) we keep only zeroth and first order terms in  $\alpha'$ .

The scalar field  $\phi$  is written like  $\phi = \phi^{(0)} + \alpha' \phi^{(1)}$ , where  $\phi^{(0)}$  is a classical zero-order solution. As *Case (b)* of Section 2 cannot represent cylindrically symmetric spacetime, we take

$$\phi^{(0)} = \phi_0 + \phi_1 z .$$

It should be mentioned that in some works [7] the additional term

$$S_{\text{GB}} = 8\pi^2 \lambda \int_{\mathcal{M}} e^{\phi} e(\mathcal{M}) ,$$

enters action (3.1), where  $e(\mathcal{M})$  is the Euler class of the manifold  $\mathcal{M}$ , in four-dimensions equal to

$$e(\mathcal{M}) = \frac{1}{8\pi^2} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) \eta ,$$

$\eta$  is the volume element.

Next analysis will be simplified if one adopts the following reasonable conditions on the electromagnetic field behaviour. First, the Lorentz force  $\mathcal{F}$  acting on the fluid particles,  $\mathcal{F} \propto *(\mathbf{u} \wedge *\mathbf{F})$ , vanishes (for the case with the non-vanishing Lorentz force but without the scalar field see [4]). Second, only longitudinal magnetic field survives. In comoving system the magnetic field 1-form is given by  $\mathbf{B} = *(\mathbf{u} \wedge \mathbf{F})$ , which together with the previous point gives

$$*(\mathbf{u} \wedge \mathbf{F}) \wedge \Theta^2 = 0 .$$

Before treating the equations of motion for the electromagnetic and gravitational fields we impose another supposition concerning the perfect fluid.

From the physical considerations it follows, because the fluid represents the fermionic matter, that it is more physically favourable if one assumes the presence of a continuous electric charge distribution throughout the spacetime with a charge current density 1-form  $\mathbf{j}$ . Nevertheless, to maintain the action (3.1) unaffected, the current density incorporation is permissible only in the case where the corresponding source term  $\mathbf{j} \wedge *\mathbf{A}$  does not act as an extra gravitational field source.

Furthermore it is natural to postulate that the fluid particles are charge carriers. Thus the current density  $\mathbf{j}$  is purely convectional,  $\mathbf{j} = \rho \mathbf{u}$ , with  $\rho(r)$  being an invariant charge density.

With the above in mind the generalized Maxwell equations take the form

$$- * d * (e^{\phi^{(0)}} \mathbf{F}) = \frac{4\pi}{\lambda} \rho \Theta^0 . \quad (3.2)$$

By virtue of the Einstein-Maxwell equations resulting from (3.1) (see below), the conditions above and (3.2) will be satisfied if

$$\mathbf{F} = B \exp(-\frac{1}{2}\phi_0 - \frac{1}{2}\phi_1 z - \gamma) \Theta^3 \wedge \Theta^1 , \quad B = \text{const} . \quad (3.3)$$

Note that

$$*d\mathbf{F} = \frac{1}{2} B \phi_1 \exp(-\phi^{(0)}/2 - 2\gamma) \Theta^0 , \quad (3.4)$$

which signals that we deal with a magnetic monopoles current. As a matter of fact it is necessary to introduce a magnetic charges current density 1-form  $\mathbf{j}_m$ . In principle there are possibilities to retain (3.4) physically admissible. Either we can expect that going to a non-abelian gauge fields will smooth out this solution, or, in the case of abelian gauge fields, it is possible to introduce  $\mathbf{j}_m$  explicitly in the action (3.1), but one has to break the general covariance to do this [9].

The metric field equations of motion following from the superstring effective action (3.1) are the Einstein equations (2.3) enriched by the electromagnetic field contribution [10]

$$8\pi \mathbf{T}_{\text{elmag}} = \lambda \frac{e^{\phi^{(0)}}}{8\pi} \Theta^\alpha \otimes *(\mathbf{F} \wedge i_\alpha * \mathbf{F} - i_\alpha \mathbf{F} \wedge * \mathbf{F})$$

to the stress-energy tensor field (2.4). Again we refer the reader to [4] for details.

One finds that one has six independent equations for totality of seven unknowns: four metric functions  $f$ ,  $l$ ,  $\gamma$ ,  $\delta$  and three physical quantities  $\phi$ ,  $\mu$ ,  $\rho$ . Again, we have one degree of freedom due to the  $r$ -coordinate rescaling possibility.

Bianchi identity in our case, provided that the scalar field equation of motion is fulfilled, is

$$\mathbf{u} \wedge *(\mu d\mathbf{u} + d(p \mathbf{u}) + \lambda \rho e^{\phi^{(0)}} \mathbf{F}) = 0 . \quad (3.5)$$

Inserting (2.1) and (3.3) into (3.5) one finds that the pressure  $p$  is constant.

The mathematical structure of the Einstein-Maxwell equations is much the same as classical theory, Section 2 (see also [4]). As a consequence, if one requires solution to be cylindrically symmetric, the scalar field becomes equal to

$$\phi = \phi_0 + \tilde{\phi}_1 z ,$$

where  $\tilde{\phi}_1$  is constant.

Resulting metric is given by

$$\begin{aligned} \mathbf{g} = dt \otimes dt + \frac{\Omega}{C} \gamma (dt \otimes d\varphi + d\varphi \otimes dt) - l^2 d\varphi \otimes d\varphi \\ - e^{2\gamma} dz \otimes dz - C^{-2} l^{-2} de^\gamma \otimes de^\gamma , \end{aligned} \quad (3.6)$$

$$l^2 = \frac{8\pi p}{C^2} e^{2\gamma} - \frac{4\Omega^2 + \tilde{\phi}_1^2 - 4\lambda B^2}{2C^2} \gamma + \nu .$$

Found solution (3.6) is very similar to the zero-order solution (2.8). The only difference appears in  $l^2$  function (2.9), namely the coefficient of the linear term in  $\gamma$  gets shifted due to the magnetic field presence.

The formula for the energy density

$$\mu = \frac{1}{4\pi} (2\Omega^2 - \lambda B^2) e^{-2\gamma} - 3p$$

has very transparent physical interpretation. A “specific” mass density  $\mu + 3p$  must be added to the magnetic energy density to balance the rotation.

As an important example of (3.6) we choose the integration constants like

$$16\pi p = -C^2 \nu = m^2, \quad \gamma = \frac{2C}{m^2} \operatorname{sh} \left( \frac{mr}{2} \right), \quad 4\Omega^2 = 4\lambda B^2 - \tilde{\phi}_1^2 + 2m^2(1 - C), \quad (3.7)$$

where  $m$  is constant. With this choice the energy density becomes equal to

$$8\pi\mu = \left( 2\lambda B^2 - \tilde{\phi}_1^2 + 2m^2[1 - C] \right) e^{-2\gamma} - \frac{3m^2}{2}. \quad (3.8)$$

In (3.7) and (3.8) the substitution  $Cm^{-2} \rightarrow C$  is assumed. Since the pressure is constant it follows that our spacetime can be reinterpreted as the charged dust solution (with the vanishing pressure) and the non-zero cosmological constant [4].

Note that if one wants at this stage to eliminate the perfect fluid contribution to the action (3.1) and consider only coupling of the electromagnetic and the dilatonic fields to gravity in a spacetime with the non-zero cosmological constant the following equality must hold

$$\mu + p = 0. \quad (3.9)$$

In this case the equations (3.8) and (3.9) imply  $C = 0$  along with  $\tilde{\phi}_1^2 = 2\lambda B^2 + m^2$ . The metric tensor is found to be

$$\begin{aligned} \mathbf{g} = \left( dt + \frac{4\Omega}{m^2} \operatorname{sh} \left( \frac{mr}{2} \right) d\varphi \right) \otimes \left( dt + \frac{4\Omega}{m^2} \operatorname{sh}^2 \left( \frac{mr}{2} \right) d\varphi \right) \\ - \frac{1}{m^2} \operatorname{sh}^2(mr) d\varphi \otimes d\varphi - dz \otimes dz - dr \otimes dr , \end{aligned} \quad (3.10)$$

with  $\Omega$  subject to  $4\Omega^2 = 2\lambda B^2 + m^2$ . The dilaton is approximately given by

$$\phi = \phi_0 + \left( \phi_1 + \frac{\lambda B^2}{m} \right) z . \quad (3.11)$$

The solution (3.10) and (3.11) manifestly describes the Gödel-type spacetime [8] and was found in [3] by another way when studying a homogeneous Gödel-type solutions. Note that in zero-order regime  $\lambda \rightarrow 0$  one has  $\phi_1^2 = m^2$  and  $4\Omega^2 = m^2$ , the latter equality immediately implying  $g_{\varphi\varphi} = \partial_\varphi \cdot \partial_\varphi \leq 0$ . Thus there are no closed timelike curves in the spacetime [2]. On the other hand in  $\alpha'$ -order framework  $g_{\varphi\varphi}$  becomes positive for sufficiently large  $r$  (because  $\lambda$  is positive). In this way the first-order correction causes the chronology violation.

## 4 Geometrical properties

In each tangent space  $T_p\mathcal{M}$  the projection tensor onto 3-dimensional subspaces  $W_p$ , orthogonal to  $\mathbf{u}$ , is given by

$$\mathbf{h} = \mathbf{g} - \mathbf{u} \otimes \mathbf{u} .$$

The tensor field  $-\mathbf{h}$  serves a positive definite metric on  $W_p$ .

For a given spacetime to be static there must exist hypersurface-orthogonal time-like Killing vector field [5]. In our case the vorticity 1-form  $\omega$  equals

$$\omega = \frac{1}{2} * (\mathbf{u} \wedge d\mathbf{u}) = \Omega dz .$$

In this way the metric (3.6) is static if and only if  $\Omega = 0$ . Geometrically, a collection  $W = \cup_p W_p$  is not involutive.

The fluid particles acceleration  $\dot{\mathbf{u}}$  is given by

$$\dot{\mathbf{u}} = - * (\mathbf{u} \wedge * d\mathbf{u}) ,$$

and vanishes for (2.1) showing that  $\mathbf{u}$  parallelly propagates along itself and fluid particles move geodesically. Namely for this reason, and because we have restricted our attention to the Lorentz force-free case, the pressure has to be constant.

The rate-of-strain tensor field is actually the extrinsic curvature  $\mathbf{K}$ , and is defined by

$$\mathbf{K} = \frac{1}{2} \mathcal{L}_{\mathbf{u}} \mathbf{h} .$$

Direct computation shows that  $\mathbf{K}$  vanishes identically for (3.6) which from the physical viewpoint means that the fluid rotates as a rigid body.

The last remark concerns the algebraic classification of the Weyl tensor field **C**. It turns out that for (3.6) there generally exist just four distinct null vectors **k** (modulo multiplying  $\mathbf{k} \rightarrow a\mathbf{k}$ ,  $a$  is constant) satisfying

$$k^\beta k^\gamma k_\lambda C_{\alpha\beta\gamma\delta} k_\sigma (\Theta^\lambda \wedge \Theta^\alpha) \otimes (\Theta^\delta \wedge \Theta^\sigma) = 0 .$$

Thus our spacetime (3.6) belongs to the type *I* according to the Petrov classification [5].

## Acknowledgments

The author is obligated to Dr Rikard von Unge for helpful discussions.

## References

- [1] J. D. Barrow and M. P. Dąbrowski *Phys.Rev. D* **58** (1998) 103502
- [2] S. W. Hawking and G. F. R. Ellis *The Large Scale Structure of the Spacetime* Cambridge University Press, Cambridge 1973
- [3] P. Kanti and C. E. Vayonakis *Phys.Rev. D* **60** (1999) 103519
- [4] P. Klepáč and J. Horský *Class. Quantum Grav.* **17** (2000) 2547
- [5] D. Kramer, H. Stephani, M. A. H. MacCallum and E. Hertl *Exact solutions of Einstein's field equations* VEB DAW, Berlin 1980
- [6] E. Radu *Class. Quantum Grav.* **15** (1998) 2743
- [7] N. E. Mavromatos and J. Rizos *Phys.Rev. D* **62** (2000) 124004
- [8] M. J. Rebouças and J. Tiomno *Phys.Rev. D* **28** (1983) 1251
- [9] J.H. Schwarz and A. Sen *Nucl.Phys. B* **411** (1994) 35
- [10] N. Straumann *General Relativity and Relativistic Astrophysic* Springer, Berlin 1984